FINAL STAGE IN TURBULENT VELOCITY FIELD

DEGENERATION IN A COFLOW

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Analytic dependences describing the evolution of axisymmetric and planar coflows in the asymptotic case of weak turbulence are obtained on the basis of a three-parameter differential model.

At this time considerable experience has been accumulated (see the monograph [1], for instance) on computation of turbulent shear flow characteristics on the basis of multiparametric differential ($\overline{u_i u_j}$ — $\varepsilon_u$) models. The recommended values of the empirical constants here refer to domains of strong turbulence governed by large values of the turbulent Reynolds number $R_\lambda$. In the majority of free turbulent flows this parameter $R_\lambda$ diminishes monotonically downstream from values $R_\lambda \gg 1$ in the near domain to $R_\lambda < 1$ in the far. In computing such flows on the basis of asymptotic models (for $R_\lambda \gg 1$) ever-increasing differences are detected between the computation results and experimental data as the turbulent number $R_\lambda$ diminishes. It is shown in [2, 3] that to eliminate these differences that occur, the empirical constants recommended in [1] should be replaced by functions of the number $R_\lambda$. These functions can be considered constants only in the limit cases of $R_\lambda \to \infty$ and $R_\lambda \to 0$; however, their asymptotic values as $R_\lambda \to 0$ require refinement.

The singularities of degeneration of an inhomogeneous velocity field were first investigated in the asymptotic case of weak turbulence by Phillips [4] by applying the Fourier transform to the Navier—Stokes equations for the velocity components and the vorticity with subsequent decomposition of the Fourier transforms into series and neglecting higher-order infinitesimals. It is shown in this paper that a three-parameter differential model $q^2$ — $\overline{u_1 u_2}$ — $\varepsilon_u$, describing the evolution of the far wake ($R_\lambda \to 0$), allows of analytical solution. The rate of degeneration of the wake characteristics being modeled evidently depends on the magnitude of the empirical parameters in the model. Therefore, by comparing the analytic solution obtained with the known Phillips laws, the magnitude of the empirical parameters can be determined as $R_\lambda \to 0$. Moreover, the analytic solutions can turn out to be useful in numerical modeling of the wake characteristics on the basis of a universal model with respect to the turbulent Reynolds number. Actually, the emergence of the numerical solution into an analytic solution as the number $R_\lambda$ diminishes will indicate the correctness of the numerical integration method and the absence of errors in the calculation program.

Using the diameter of a body of revolution or the transverse dimension of a flat body d and the free stream velocity $U_\infty$ as characteristic quantities, we introduce the dimensionless parameters

$$U = \frac{U_1 - U_\infty}{U_\infty}, \quad E = \frac{\overline{u_i u_i}}{U_\infty^2}, \quad x = \frac{x_1}{d}, \quad r = \frac{x_2}{d},$$

$$R = \frac{\overline{u_1 u_2}}{r U_\infty^2}, \quad D = \varepsilon_u \frac{d}{U_\infty^3}, \quad R_\infty = \frac{U_\infty d}{\nu}.$$

For $R_\lambda < 1$ the inertial forces become negligible compared with the viscous forces; consequently, the closed system of equations (known from [1-3], say) and the condition for conservation of the excess momentum are simplified and take the form

$$\frac{\partial U}{\partial x} = \frac{1}{r^n R_\infty} \frac{\partial}{\partial r}\left(r^n \frac{\partial U}{\partial r}\right), \quad I_u = \int_0^\infty r^n U dr = \text{const}, \tag{1}$$

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$$\frac{\partial R}{\partial x} = \frac{1}{rR_\infty}\frac{\partial}{\partial r}\left(r^n\frac{\partial R}{\partial r}\right) + \frac{2}{rR_\infty}\frac{\partial R}{\partial r} - $$
$$- c_1\frac{E}{r}\frac{\partial U}{\partial r} - c_2\frac{D}{E}R, \quad (2)$$
$$\frac{\partial E}{\partial x} = \frac{1}{r^n R_\infty}\frac{\partial}{\partial r}\left(r^n\frac{\partial E}{\partial r}\right) - 2D,$$
$$\frac{\partial D}{\partial x} = \frac{1}{r^n R_\infty}\frac{\partial}{\partial r}\left(r^n\frac{\partial D}{\partial r}\right) - F_u\frac{D^2}{E}. \quad (3)$$

The value n = 0 corresponds to plane flow, and n = 1 to axisymmetric flow.

We assume a self-similar nature to the dependence of the characteristics being modeled on the coordinates x and r. Following the method elucidated by Gorodtsov [5], it is easy to obtain

$$U(x,\ \eta) = U_0(x+x_0)^{nu}f_u(\eta),\quad E(x,\ \eta) = E_0(x+x_0)^{nE}f_E(\eta),$$
$$D(x,\ \eta) = D_0(x+x_0)^{nD}f_D(\eta),\quad R(x,\ \eta) = R_0(x+x_0)^{nR}f_R(\eta),$$
$$\eta = r(x+x_0)^{-0,5}.$$

The damping exponents and the transverse coordinate functions are determined from the system of ordinary differential equations

$$\eta\cdot nu\cdot f_u - \frac{\eta^2}{2}f_u' = \frac{\eta^{1-n}}{R_\infty}(\eta^n f_u')', \quad (4)$$
$$\eta\cdot nE\cdot f_E - \frac{\eta^2}{2}f_E' = \frac{\eta^{1-n}}{R_\infty}(\eta^n f_E')' - \frac{2\eta D_0}{E_0}f_D, \quad (5)$$
$$\eta\cdot nD\cdot f_D - \frac{\eta^2}{2}f_D' = \frac{\eta^{1-n}}{R_\infty}(\eta^n f_D')' - \frac{\eta F_u D_0}{E_0}\frac{f_D^2}{f_E}, \quad (6)$$
$$\eta\cdot nR\cdot f_R - \frac{\eta^2}{2}f_R' = \frac{\eta^{1-n}}{R_\infty}(\eta^n f_R')' + \frac{2}{R_\infty}f_R' - \frac{\eta c_2 D_0}{E_0}\frac{f_D f_R}{f_E} - c_1\frac{U_0E_0}{R_0}f_E f_u', \quad (7)$$
$$I_u = \int_0^\infty \eta^n f_u(\eta)\,d\eta = \text{const}, \quad (8)$$

where

$$nu > nR + 1,\ nE > nR + nu + 1. \quad (9)$$

Each of the functions should satisfy the evident boundary conditions

$$f'|_{\eta=0} = 0,\quad \lim_{\eta\to\infty} f'(\eta) = \lim_{\eta\to\infty} f(\eta) = 0.$$

From (5) and (6) we obtain

$$f_D = \frac{f_E}{F_u - 2},\quad nD = nE - 1,\quad nE = \frac{(F_u-2)n + F_u + 2}{2(2-F_u)}. \quad (10)$$

In a planar wake with nonzero excess impulse the turbulent Reynolds number is conserved invariant downstream and there is no final stage of degeneration. In a planar momentum-free medium and in an axisymmetric medium (for any magnitude of the excess momentum $I_u$) the number $R_\lambda$ diminishes to zero. Let us examine in greater detail the axisymmetric wave as $R_\lambda \to 0$. Here $nE = F_u/(2 - F_u)$ independently of the quantity $I_u$. The limit value of the function $F_u$, equal to 2.8, is determined in [2] from the invariant relationship of L. G. Loitsyanskii. Therefore, nE = —3.5, i.e., agrees with the exponent obtained by Phillips [4].

The equation for the function $f_E$ takes the form

$$(\eta f_E')' + \alpha\eta^2 f_E' + 2\alpha\eta f_E = 0,\quad R_\infty = 2\alpha.$$

The substitution $z = \eta^2$ converts it into an equation whose general solution is known [6]. By satisfying the boundary conditions we obtain

$$E(x, \eta) = E_0 (x + x_0)^{-3,5}\Phi(\eta), \quad D(x, \eta) = 1{,}25\frac{E(x, \eta)}{x + x_0},$$

$$\Phi(\eta) = \exp\left(-\frac{\alpha\eta^2}{2}\right).$$

Therefore, the Taylor microscale $\lambda_u$ grows according to the law $\sqrt{x + x_0}$ and remains constant across the wave in the final stage of axisymmetric wave degeneration.

Under the condition $I_u = 0$, there follows uniquely $nu = -1$ from (4) and (8). The function $f_u$ is described by (4), and, therefore, $U(x, \eta) = U_0(x + x_0)^{-1}\Phi(\eta)$.

In the case of zero excess momentum, the integral condition (8) does not permit determination of the exponent nu. As follows from [6], the solution of the differential equation (4) that satisfies the symmetry condition can be represented in the form

$$f_u = \Phi(\eta)_1F_1\left(nu - 1, \quad 1; \quad \frac{\alpha\eta^2}{2}\right),$$

where ${}_1F_1(a, b; x) = \sum\limits_{k=0}^{\infty}\frac{(a)_k x^k}{(b)_k k!}$ is a degenerate hypergeometric function. It is natural to assume that as in the case $I_u \neq 0$ the defect in the velocity in a momentum-free wake will damp out exponentially as $\eta \to \infty$. This is possible only in the case that the exponent nu is an arbitrary negative integer where $nu \leqslant -2$ . A definite function $f_u$ corresponds to each value of the exponent nu, for instance

$$nu = -2, \quad f_{u1}(\eta) = \Phi(\eta)\left(1 - \frac{\alpha}{2}\eta^2\right), \tag{11}$$

$$nu = -3, \quad f_{u2}(\eta) = \Phi(\eta)\left(1 - \alpha\eta^2 + \frac{\alpha^2}{8}\eta^4\right). \tag{12}$$

From relationship (9) there follows $1 \leqslant nu - nR < -4.5$. Consequently, to determine the possible values of the exponent nu it is necessary to find nR. In the case of a momentum-free wake, the component $(E/r)\partial U/\partial r$, modeling the generation of the tangential stresses in (2) decreases according to a power law with the exponent $n = -4.5 + nu \leqslant -6.5$. We assume that the contribution of this component to the balance equation can be neglected. Then, multiplying (7) term-by-term by $\eta^2$ and integrating over the transverse coordinate, we have

$$(nR + 1{,}25c_2 + 2)\int\limits_0^\infty \eta^3 f_R(\eta)\,d\eta = 0, \quad nR = -2 - 1{,}25c_2.$$

It is known that $c_2 = 2 + c_2'$, where $c_2'$ enters in the exchange approximation and the factor 2 in the approximation of the dissipative components in the second-moment balance equations. Each of the pulsation components dissipates the stored energy without exchange with the other components in the final stage of the degeneration. Therefore, $c_2' \to 0$, $c_2 \to 2$, and $nR = -4.5$. In such a case the convective and dissipative terms of (2) decrease according to a power law with exponent $n = -5.5$, and the contribution of the generation to the balance equations can actually be neglected.

It is easy to see that for $I_u \neq 0$ all the components of (2) are equally right and decrease according to the law $(x + x_0)^{-5.5}$. Therefore, the exponent $nR = -4.5$ is independent of the magnitude of the excess momentum, while the function $f_R$ is described by different equations for $I_u = 0$ and $I_u \neq 0$. The model system of equations (1)-(3) hence allows power laws of rate defect damping with both the factor $nu = -2$, and with the subscript $nu = -3$. The distribution $f_u$ across the wake is described by relationships (11) and (12), and $R(x, \eta) = R_0(x + x_0)^{-4.5}\Phi(\eta)$.

For $I_u \neq 0$ the function $f_R$ satisfies the equation

$$(\eta f_R')' + (2 + \alpha\eta^2) f_R' + 4\alpha\eta f_R = A\eta\Phi^2(\eta), \quad A = -\frac{\alpha^2 c_1 U_0 E_0}{R_0},$$

whose solution has the form

$$f_R(\eta) = \Phi(\eta) + \frac{A}{2\alpha^2}\Phi(\eta)\int\limits_0^\eta [\Phi^{-1}(t) - (1 + \alpha t^2)\Phi(t)]\,t^{-3}\,dt.$$

For a flat momentum-free wake, $n_E = -3$, $f_D = 1.25\, f_E$ follows from (10). The function $f_E$ satisfies the equation

$$f''_E + \alpha\eta f'_E + \alpha f_E = 0,$$

that has the solution $f_E(\eta) = \Phi(\eta)$. Therefore, in the final stage of degeneration of a planar momentum-free turbulent wake, the turbulent energy and its dissipation rate satisfy the relationships

$$E(x,\ \eta) = E_0(x + x_0)^{-3}\Phi(\eta),\quad D(x,\ \eta) = 1{,}25\frac{E(x,\ \eta)}{x + x_0},$$

and the Taylor microscale $\lambda_u$ across the wake is kept constant. Multiplying (7) by $\eta^2$ and integrating with respect to the transverse coordinate with the boundary conditions taken into account, we obtain

$$(nR + 1{,}25c_2 + 1{,}5)\int_0^\infty \eta^2 f_R(\eta)\, d\eta = 0.$$

Therefore, $nR = -1.5 - 1.25c_2 = -4$, and (7) is converted to the form $(\eta^2 f'_R)' + \alpha(\eta^3 f_R)' = 0$ and allows of analytic solution so that $R(x, \eta) = R_0(x + x_0)^{-4}\Phi(\eta)$.

Writing the general solution of (4) by analogy with the axisymmetric case and assuming an exponential nature of the decrease in the velocity defect as $\eta \to \infty$, we easily obtain that $nu = -k - 1/2$, $k \leqslant 0$ is an arbitrary integer. But in the final stage of degeneration of a plane momentum-free wake the exponent nu should satisfy inequalities (9). In that case the model (1)-(3) allows two laws of evolution for the defect of the average velocity

$$U(x,\ \eta) = U_{10}(x + x_0)^{-1,5}(1 - \alpha\eta^2)\Phi(\eta), \tag{13}$$

$$U(x,\ \eta) = U_{20}(x + x_0)^{-2,5}\left(1 - 2\alpha\eta^2 + \frac{1}{3}\alpha^2\eta^4\right)\varphi(\eta). \tag{14}$$

In conclusion, we note that because of the linearity of the equation for the defect in the mean velocity, its solution will generally be a linear combination of the functions (13) and (14). Consequently, in both the plane and axisymmetric flows the defect in the mean velocity will be described by functions that damp more slowly for sufficiently large values of the longitudinal coordinate.

## NOTATION

$u_i$, velocity fluctuation components; $q^2 = u_i u_i$, doubled kinetic energy of the velocity fluctuations; $\varepsilon_u = \nu(\partial u_i/\partial x_k)^2$, turbulence kinetic energy dissipation rate; $\lambda_u = \sqrt{5\nu q^2/\varepsilon_u}$, Taylor microscale; $R_\lambda = q\lambda/\nu$, turbulent Reynolds number; $x$, $\eta$, longitudinal and transverse self-similar coordinates; nu, nE, nD, nR, exponents of the damping power laws; $f_u$, $f_E$, $f_D$, $f_R$, self-similar profile functions; $F_u$, $c_1$, $c_2$, empirical coefficients.

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